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ON THE THEORY OF THE SLIPPING STATE IN DYNAMICAL SYSTEMS WITH COLLISIONS

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For a wide class of systems with collisions we propose an approximate method for computing the slipping states, characterized by the same degree of completeness and labor-consumption as the known methods for computing motions of simple types. For the case when the relative acceleration of the colliding bodies varies by a linear law on the final segment of the slipping state, we have obtained an analytic expression for the state's duration factor as a function of the velocity recovery factor under impact.

Examples of the calculation of concrete models are considered. A comparison with results obtained by exact methods shows that the error does not exceed a few percents even for the first approximation. By a slipping state in a system with collisions we mean a motion accompanied on a finite time interval by an infinite sequence of instantaneous shock interactions between two fixed elements of the system. For a wide class of systems being considered the problem has been solved in [1 – 3] of determining in phase space the exact boundaries of the slipping state regions and of delineating the existence regions of periodic motions with a slipping state segment in parameter space. However, it is not advisable to recommend the use of the iterative procedure used in those papers as a practical calculation method because of the considerable consumption of labor. The approximate method of analyzing slipping states was suggested in [4]. But, as was noted in [5, 6], when it was applied to concrete models, significant segments of the functions being computed could not be obtained successfully.

In the present paper we propose an approximate method for calculating the slipping state regions in phase space and the existence regions of periodic motions with a slipping state segment in parameter space.

The method is based on an analysis of point transformations of the shock interaction hyperplane into itself and on the following two idealizations: a) a certain new characteristic, which we introduce into consideration, namely, the duration factor of the slipping state, which depends only on the physical parameters of the system; b) the slipping state starting at some instant can also be treated as the motion of colliding masses with a superimposed kinematic constraint after their absolutely inelastic interaction [1]. In such an approach the approximateness of the method is connected with the approximateness of the idealizations adopted. However, the possibility remains of an unbounded refinement of the dynamic model being analyzed at the expense of choice of the instant taken as the start of the slipping state.

1. Equation of the slipping state region in phase space. Let us consider a dynamic system whose dimensionless equations of motion in the intervals between collisions can be represented in the form

$$x_1^{\cdot\cdot} = F_1 + F_{12}, \qquad \mu x_2^{\cdot\cdot} = F_2 - F_{12}$$
$$x_i^{\cdot\cdot} = F_i \qquad (i = 3, 4, \ldots, n), \qquad x_2 - x_1 > 0 \tag{1.1}$$

Here x_1 and x_2 denote the displacements of the colliding masses, where the origin of reference is chosen so that the impact takes place when the equality $x_2 = x_1$ is fulfilled; μ is the ratio of the colliding masses, while F_{12} is the interaction force between them, depending only on the difference $x_2 - x_1$. We assume that the forces F_i $(i = 1, 2, \ldots, n)$ do not depend upon the quantities $x_1^{\cdot}$ and $x_2^{\cdot}$ and are analytic functions of the remaining phase coordinates $x_1, \ldots, x_n, x_3^{\cdot}, \ldots, x_n^{\cdot}, t$. Thus, the right-hand sides of system (1.1) retain continuity at the collision instants.

We make a change of two variables by the formulas $y = x_2 - x_1$, $y^{\cdot} = x_2^{\cdot} - x_1^{\cdot}$. Then the equations of motion (1.1) take the form

$$\mu y^{\cdot\cdot} = F_2 - \mu F_1 - (1 + \mu) F_{12}, \qquad (1 + \mu) x_1^{\cdot\cdot} = F_1 + F_2 - \mu y^{\cdot\cdot}$$
$$x_i^{\cdot\cdot} = F_i \qquad (i = 3, 4, \ldots n), \qquad y > 0 \tag{1.2}$$

By idealizing the impact as being instantaneous with a velocity recovery factor $0 \leqslant R < 1$, we have the known relations between the post-collision $y^{\cdot +}$, $x_1^{\cdot +}$ and the pre-collision $y^{\cdot -}$, $x_1^{\cdot -}$ values of the velocities $y^{\cdot}$, $x_1^{\cdot}$ in the form

$$y^{\cdot +} = - R y^{\cdot -}, \qquad x_1^{\cdot +} = x_1^{\cdot -} + \frac{\mu (1 + R)}{1 + \mu} y^{\cdot -} \tag{1.3}$$

In the phase space $x_1, y, x_3, \ldots, x_n, x_1^{\cdot}, y^{\cdot}, x_3^{\cdot}, \ldots x_n^{\cdot}, t$ of the system being analyzed the shock interactions take place on a hyperplane Π described by the relation

$y = 0$. The point transformation $M_1 = T(M_0)$, generated by Eqs. (1.2), (1.3), maps the surface Π into itself. The slipping state region Π_s is the part of Π contained between the manifold $y^{\cdot} = 0$ and some boundary Γ_s. The latter separates the system's states which are the initial ones for subsequent slipping state from the initial conditions for which a slipping state does not occur during the motion. It is necessary to note that knowledge of only a segment $\Gamma_s' \in \Gamma_s$ in an arbitrarily small neighborhood of the exit manifold M_s from region Π_s permits us to extend, by a consideration of the inverse transformation $T^{-1}(\Gamma_s')$, the known part of the boundary Γ_s from Γ_s' up to Γ_s'', Γ_s''', ... [2, 3]. Here each time only the number of the collision which can be taken as the initial one for a subsequent slipping state, is displaced. Therefore, without loss of generality, it is sufficient to delineate the most important part of Γ_s, abutting manifold M_s.

From the analysis of the point transformation $M_1 = T(M_0)$ it is always possible to obtain (even if by a numerical method) [7] a certain relation

$$\varphi(\tau_0, x_{10}, 0, x_{30}, \ldots, x_{n0}, x_{10}^{\cdot}, y_0^{\cdot}, x_{30}^{\cdot}, \ldots, x_{n0}^{\cdot}, t_0) = 0 \tag{1.4}$$

connecting the duration $\tau_0 = t_1 - t_0$ between the initial and the subsequent collisions with the coordinates of the initial point

$$M_0(x_{10}, 0, x_{30}, \ldots, x_{n0}, x_{10}^{\cdot}, y_0^{\cdot}, x_{30}^{\cdot}, \ldots, x_{n0}^{\cdot}, t_0)$$

The duration τ_0 is the smallest positive simple root of Eq. (1.4). The continuous dependence of τ_0 on the coordinates of the initial point is violated when this quantity degenerates into a multiple root of Eq. (1.4), i.e. on some manifold Γ_1 defined by the conditions

$$y = 0, \quad \varphi = 0, \quad \varphi_{\tau_0}' = 0 \tag{1.5}$$

The feature of the trajectories issuing from Γ_1 at the instant t_0 is that at the instant $t_0 + \tau_0$ they are only tangent to hyperplane Π. Therefore, the equations of Γ_1 can be obtained also as the inverse point transformation T^{-1} of the manifold $y = 0$, $y^{\cdot} = 0$.

We introduce into consideration a certain new characteristic. We give the name duration factor Θ of the slipping state to the ratio of the time interval τ_0 to the total duration h of the slipping state starting at the instant t_0 and terminating at the instant t_s on the exit manifold, i.e.

$$\Theta = \tau_0 h^{-1} \tag{1.6}$$

We assume that the factor Θ does not depend on the phase coordinates and is characterized only by the system parameters. Then Eqs. (1.4), (1.6) permit us to define, formally, on the hyperplane Π certain manifolds W_h corresponding to specific values $h = \text{const}$. These manifolds form a one-parameter family located between the boundary $y^{\cdot} = 0$ and the envelope (1.5) of family (1.4). It is obvious that with due regard to the assumptions we have accepted on the nature of factor Θ, only those points of manifolds W_h have a physical meaning, which are simultaneously located on boundary Γ_s. To single out these points we make use of a method, independent of the preceding considerations, for determining the duration h of the slipping state.

We know that when system (1.2) and (1.3) moves in a slipping state, the phase trajectory corresponding to it, starting with some instant t_0, is arbitrarily close to the phase trajectory of the idealized system with a superimposed kinematic constraint between the colliding masses [1]. We obtain the law of motion of this dynamic model from relations (1.2), by replacing the first two of them by the equations of motion of the interacting masses in a kinematic constraint

$$G(t) \equiv F_2 - \mu F_1 - (1 + \mu)F_{12} < 0, \quad (1 + \mu) x_1^{\cdot\cdot} = F_1 + F_2 \tag{1.7}$$
$$x_i^{\cdot\cdot} = F_i \quad (i = 3, 4, \ldots, n), \quad y = 0$$

We obtain the equations of the exit manifold M_s from the condition of termination of motion in a kinematic constraint

$$y = 0, \;\; y^{\cdot} = 0, \quad G(t) = 0, \;\; G^{\cdot}(t) > 0 \tag{1.8}$$

Relations (1.7), (1.8) permit us to obtain, approximately from an analysis of the motion with the superimposed constraint, after an absolutely inelastic impact, the dependence of the slipping state's duration on the coordinates of the initial point of the transformation in a form analogous to expression (1.4), i.e.

$$\Phi(h, x_{10}, \ldots, x_{n0}, x_{10}^{\cdot}, \ldots, x_{n0}^{\cdot}, t_0) = 0 \tag{1.9}$$

The quantity h is determined as the smallest positive root of Eq. (1.9). The one-parameter family defined by relation (1.9) consists of manifolds V_h located on surface Π, each of which corresponds to a specific duration h = const of the motion of the colliding masses in a kinematic constraint. The system of Eqs. (1.4), (1.6), (1.9) impose an additional constraint on the coordinates of hyperplane Π and, by the same token, define the desired boundary Γ_s as the locus of the intersections of manifolds W_h and V_h for various values of parameter h. Examples of such intersections are shown in Fig. 2, Sect. 3.

It should be noted that the delineation of the boundary of region Π_s can be looked upon as a problem of obtaining sufficient conditions for the structural stability [coarseness (*)] of the parameter space [8] of a system with collisions with respect to the replacement of the real values of factor R by zero. If for $R \neq 0$ the system's state at some instant t_0 is characterized by a phase point $M_0 \in \Pi_s$, then the idealization of the subsequent behavior (the slipping state) as the motion of interacting masses with a superimposed kinematic constraint is, in principle, permissible. The errors in such an idealization can be made arbitrarily small at the expense of choosing t_0, namely, "its starting instant", since here the corresponding phase trajectories are arbitrarily close. However, in case $M_0 \overline{\in} \Pi_s$, the stated idealization is unjustified because the phase trajectories issuing from the point M_0 for a system with $R \neq 0$ and a system with $R = 0$, in general, do not come together unboundedly on a given finite interval.

After the equations for boundary Γ_s have been obtained, the study of periodic motions with a slipping state section reduces to the usual procedure of seeking the fixed points of the corresponding point transformations of surface Π into itself and to the investigation of their stability. The constraint $M_0 \in \Pi_s$, imposed on one of the fixed points M_0, is an additional requirement. By replacing this condition by the relation $M_0 \in \Gamma_s$, from the system for determining the coordinates of the fixed points we obtain the equations of the boundary C_s of the existence regions of the motions being considered in the space of the parameters of the dynamic system.

2. Duration factor of the slipping state under a linear law of

*) Editor's Note. See PMM Vol. 33, №6, 1969, page 948 (English version).

variation of the relative acceleration of the colliding bodies. Let us show that in a wide and, seemingly, most important class of dynamic systems with collisions the factor Θ is a comparatively simple function of the velocity recovery factor R. We consider the case when the relative acceleration $y^{\cdot\cdot}(t)$ in a neighborhood $[t_0, t_0 + h]$ of the instant t_0, taken as the starting time of the slipping state, can be taken approximately as a linear time function of the form

$$y^{\cdot\cdot}(t) = y^{\cdot\cdot}(t_0) + (t - t_0)Y \tag{2.1}$$

Then, on the interval (t_k, t_{k+1}) the law of variation of the relative distance $y(t)$ can be represented by Taylor's formula

$$y(t) = -Ry_k^{\cdot}(t - t_k) + \frac{(t - t_k)^2}{2} y_k^{\cdot\cdot} + \frac{(t - t_k)^3}{6} Y$$
$$(y_k^{\cdot} = y^{\cdot -}(t_k), \quad y_k^{\cdot\cdot} = y^{\cdot\cdot}(t_k)) \tag{2.2}$$

We assume that the necessary conditions for the existence of a slipping state are fulfilled, i.e. $y_k^{\cdot} < 0$, $y_k^{\cdot\cdot} < 0$, $Y > 0$ [1].

The equality $y(t_{k+1}) = 0$ holds at the instant t_{k+1}. Therefore, for determining the interval $\tau_k = t_{k+1} - t_k$ between two successive collisions we have, from formula (2.2), the following relation:

$$-Ry_k^{\cdot} + {}^1/_2\tau_k y_k^{\cdot\cdot} + {}^1/_6\tau_k^2 Y = 0 \tag{2.3}$$

where τ_k is the smallest positive root of Eq. (2.3), i.e.

$$\tau_k = -\frac{3y_k^{\cdot\cdot}}{2Y}\left[1 - \left(1 - \frac{2}{3}R\alpha_k\right)^{1/2}\right], \qquad \alpha_k = -4Y\frac{y_k^{\cdot}}{(y_k^{\cdot\cdot})^2} \tag{2.4}$$

When the sufficient existence conditions for the slipping state [1] are fulfilled, and, in particular, the inequality
$$\frac{5(R-1)}{6R} < \frac{4Yy_0^{\cdot}}{(y_0^{\cdot\cdot})^2}$$
the quantity $|{}^2/_3 R\alpha_k| < 1$ and the right-hand side of equality (2.4) can be represented by an absolutely convergent series in positive integral powers of the product $R\alpha_k$

$$\tau_k = 2R\frac{y_k^{\cdot}}{y_k^{\cdot\cdot}}\left[1 + 2\sum_{n=1}^{\infty}\frac{(2n-1)!}{6^n (n-1)!\,(n+1)!}(R\alpha_k)^n\right] \tag{2.5}$$

In accordance with (2.2), for the determination of the quantities $y_k^{\cdot}$ and $y_k^{\cdot\cdot}$ we have, from (2.5), the following relations

$$y_{k+1}^{\cdot} = Ry_k^{\cdot}[1 - {}^1/_6R\alpha_k - {}^1/_{18}(R\alpha_k)^2 - \ldots]$$
$$y_{k+1}^{\cdot\cdot} = y_k^{\cdot\cdot}[1 - {}^1/_2R\alpha_k - {}^1/_{12}(R\alpha_k)^2 - \ldots] \tag{2.6}$$

Using expressions (2.5), (2.6) as iteration transformations, we express τ_k in terms of the quantities $y_0^{\cdot}$, $y_0^{\cdot\cdot}$. As a result we obtain

$$\tau_k = 2R^{k+1}\frac{y_0^{\cdot}}{y_0^{\cdot\cdot}}\left\{1 + \left(2\frac{1-R^k}{1-R} + R^k\right)\frac{R\alpha_0}{6} + \left[7\frac{1-R^{2k}}{1-R^2} + 2R^{2k} + \frac{7R}{1-R}\left(2\frac{1-R^{2k-2}}{1-R^2} - 2R^{k-1}\frac{1-R^{k-1}}{1-R} + R^{k-1} - R^{2k-1}\right)\right]\frac{(R\alpha_0)^2}{36} + \ldots\right\}$$

Summing the sequence $\{\tau_k\}$ $(k = 0, 1, 2, \ldots)$, we obtain total duration of the interval, corresponding to the infinitely colliding process, in the form of series

$$h = \sum_{k=0}^{\infty} \tau_k = \frac{2Ry_0^{\cdot}}{(1-R)\,y_0^{\cdot\cdot}} \left\{1 + \frac{R\alpha_0}{6(1-R)}\left[1 + \frac{R\alpha_0}{6(1-R^3)}(2 + 3R + 2R^2)\right]\ldots\right\} \tag{2.7}$$

We now proceed to obtain the dependency of the quantity τ_0 on the duration h. According to (2.5) we have

$$\tau_0 = -\frac{R\alpha_0 y_0^{\cdot\cdot}}{2Y}\left[1 + 2\sum_{n=1}^{\infty} \frac{(2n-1)!}{6^n (n-1)!\,(n+1)!}(R\alpha_0)^n\right] \tag{2.8}$$

Passing to the limit in expression (2.1) as $k \to \infty$ leads, in the boundary case of termination of the slipping state on the exit manifold, to the equation

$$y_0^{\cdot\cdot} + hY = 0 \tag{2.9}$$

Introducing the dimensionless parameter

$$\delta = \frac{R\alpha_0}{1-R} \tag{2.10}$$

we obtain from (2.7) – (2.10) the following system of equations:

$$\Theta = \frac{1-R}{2}\,\delta\left[1 + 2\sum_{n=1}^{\infty}\frac{(2n-1)!}{6^n (n-1)!\,(n+1)!}(1-R)^n \delta^n\right]$$

$$\delta\left[1 + 2\sum_{n=1}^{\infty}\frac{(2n-1)!}{6^n (n-1)!\,(n+1)!}\,\delta^n f_n(R)\right] = 2 \tag{2.11}$$

$$f_1(R) = 1, \qquad f_2(R) = \frac{2 + 3R + 2R^2}{2(1 + R + R^2)}$$

$$f_3(R) = \frac{5 + 18R + 19R^2 + 19R^3 + 18R^4 + 5R^5}{5(1 + R + R^2)(1 + R + R^2 + R^3)}, \ldots, f_n(0) = 1$$

for determining the duration factor Θ of the slipping state. From (2.11) it follows immediately that Θ depends only on the velocity recovery factor R. The accuracy of computation of factor Θ and of the quantity δ depends on the number of terms taken into account in series (2.5) and, respectively, on the number of terms in series (2.11). We note that the parameter δ depends weakly on R as R varies in the interval [0, 1] the zeroth δ_0 and the first δ_1 approximations of the parameter δ remain constant ($\delta_0 = 2$, $\delta_1 = 1.584$), $1.464 \leqslant \delta_2 \leqslant 1.448$ (sic) (*), $1.386 \leqslant \delta_3 \leqslant 1.414$. In the particular case $y^{\cdot\cdot}(t) =$ const, it follows from (2.7), (2.8) that the factor Θ coincides with its own zeroth approximation $\Theta_0 = 1 - R$. Various approximations Θ_n of factor Θ are shown in Fig. 1.

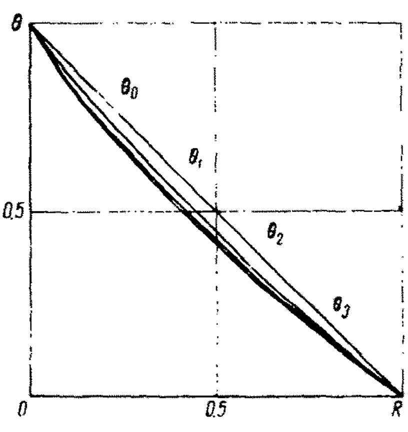

Fig. 1

As direct calculations have shown, the third

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*) Editor's Note. This is an obvious misprint in the Russian original p. 845. There is no way to correct it.

approximation Θ_3 turns out to be completely sufficient for practical purposes, i.e. only the first four terms in series (2.11) need be taken into account. In this case, to describe the duration factor Θ of the slipping state it is appropriate to use an interpolating parabolic function of the form

$$\Theta(R) = (1 - R)(1 - 0.4R) \tag{2.12}$$

As R varies over the interval $0 \leqslant R \leqslant 0{,}6$, most important in practice, the discrepancy between the values of factor Θ determined from relations (2.11) and (2.12) does not exceed 2.5%.

3. Examples. 1. From the equations for the collisionless motions of a simplest two-mass system with two colliding pairs [9]: $y^{\cdot\cdot} = -\sin t$, $|y| < d$, for the colliding interactions: $y^{\cdot+} = -Ry^{\cdot-}$, $|y| = d$, for the state of the kinematic constraint: $y = +d$ $(y = -d)$, $\sin t > 0$, and for the conditions (1.8) of termination of this motion, follows $\varphi(\tau_0, y_0^{\cdot}, t_0) \equiv \sin(t_0 + \tau_0) - \sin t_0 - \tau_0(\cos t_0 + Ry_0^{\cdot}) = 0$

$$\Phi(h, t_0) \equiv h + t_0 - \pi = 0 \tag{3.1}$$

With due regard to (1.6) the equation for the boundary Γ_s^- of region Π_s^-, located on the surface $y = -d$, as follows from relations (3.1), has the form

$$\sin[t_0 + \Theta(\pi - t_0)] - \sin t_0 - \Theta(\pi - t_0)(\cos t_0 + Ry_0^{\cdot}) = 0 \tag{3.2}$$

Figure 2 shows the results of the computation of boundary (3.2) for individual values of factor R. The dotted lines depict the refined boundaries of region Π_s^-, obtained by iterating the inverse point mappings [2]. From a comparison it follows that even the first approximation has a scarcely noticeable discrepancy with the refined solution only in the vicinity of the values, largest in modulus, of the pre-collision velocities. An analogous region Π_s^+ is located on the surface $y = +d$.

In accordance with the general approach presented in Sect. 1, from the condition that the first point of intersection with surface $y = +d$ of the phase trajectory passing through the the exit point $M_s^-(d, 0, \pi)$ from region Π_s^- lies on the boundary (3.2), we have the boundary C_s of the existence region of the elementary periodic motions with one segment of slipping state as the half-period in the following parametric form:

$$\begin{gathered}\sin[R\lambda(1.4 - 0.4R)] + \lambda(1 - 1.4R + 0.4R^2)[(1 + R)\cos\lambda + R] - \sin\lambda = 0 \\ 2d = \pi - \lambda + \sin\lambda, \quad |\lambda| \leqslant \pi\end{gathered} \tag{3.3}$$

Direct computations showed that the discrepancy between the locations of boundary C_s, constructed by the iterative procedure [2] and by relations (3.3), does not exceed 2%.

2. The motions of a one-mass model of a vibrating striker are described by the equations [3, 10]

$$y^{\cdot\cdot} = \sin t - \lambda^2(y + d), \quad y > 0; \quad y^{\cdot+} = -Ry^{\cdot-}, \quad y = 0 \tag{3.4}$$

An analysis of the point mappings, generated by relations (3.4), of surface Π into itself leads to the following dependency:

$$\begin{gathered}\varphi(\tau_0, y_0, t_0) \equiv (\alpha\cos\tau - Ry_0^{\cdot})\sin 2\lambda\tau_0 + 2\lambda[d - \alpha\sin t_0 + \\ (\alpha\sin\tau - d)(\sin^2\lambda\tau_0 + \cos\lambda\tau_0)] = 0 \\ \alpha = (\lambda^2 - 1)^{-1}, \qquad \tau = t_0 + \tau_0\end{gathered} \tag{3.5}$$

From the equations $y = 0$, $\sin t - \lambda^2 d < 0$ of the kinematic constraint between the mass and the barrier we obtain the expression

$$\Phi(h, t_0) \equiv h + t_0 - \arcsin(\lambda^2 d) \tag{3.6}$$

which, together with conditions (2.12), (3.5), determines in the first approximation the boundary Γ_s. Examples of the computation of Γ_s by the equations obtained are presented in Fig. 3. The curves 1 — 4 correspond to the following parameter values:

Curve 1 — $\lambda = 0.3$, $d = -5.0$, $R = 0.3$,
" 2 — $\lambda = 0.3$, $d = 0.5$, $R = 0.4$,
" 3 — $\lambda = 0.5$, $d = 0$, $R = 0.3$,
" 4 — $\lambda = 0.9$, $d = 0.2$, $R = 0.3$.

For comparison the dotted lines show the refined boundaries of region C_s obtained by the iterative procedure [2]. The insignificant discrepancy between the results attests not only to the admissibility of the idealization adopted but also to the sufficiency of considering here the first approximation in a comparatively large neighborhood of the point M_s $(0, 0, \arcsin(\lambda^2 d))$.

Let us obtain the equations of boundary C_s of the existence region of the simplest periodic motions with a segment of slipping state in the space of the parameters μ, d, R. For this we write down the condition for the first point of intersection with surface Π of the phase trajectory of collisionless motions,

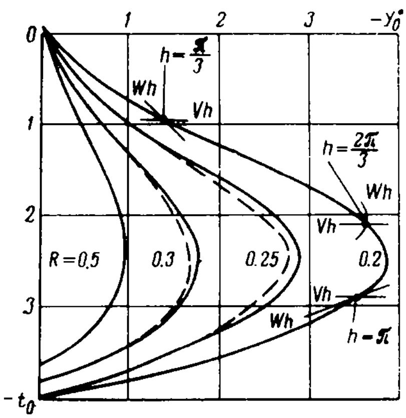

Fig. 2

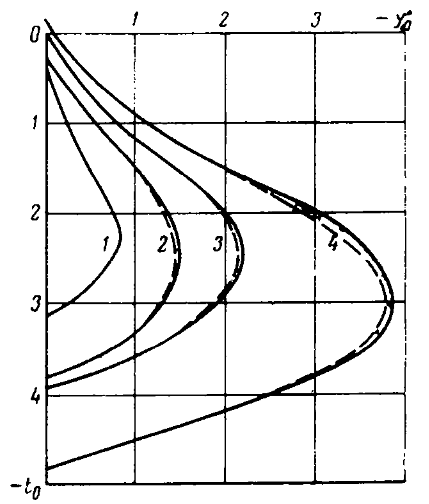

Fig. 3

issuing from the exit point M_s, to lie on the curve Γ_s:

$$\begin{gathered} d[1 + \alpha\cos\lambda(t_0 - t_s)] - \alpha\sin t_0 + \alpha\lambda^{-1}\cos t_s \sin\lambda(t_0 - t_s) = 0 \\ \alpha^{-1}y_0^{\cdot} - \lambda d\sin\lambda(t_0 - t_s) + \cos t_s\cos\lambda(t_0 - t_s) - \cos t_0 = 0 \end{gathered} \tag{3.7}$$

It is obvious that in the system of relations (2.12), (3.5) – (3.7) the variables τ_0, h, y_0 easily lend themselves to elimination, and the boundary C_s can be represented in parametric form by two transcendental equations.

3. We consider the slipping state problem under an interaction with a fixed plane of a two-mass system with an elastic constraint between the elements. A particular case of this autonomous system, when up to the instant of the first collision the masses have like velocities and move as a unit with undeformed elastic constraint,

was investigated in [5] by the method proposed in [4]. The system's motions are described by the equations

$$m_1 y^{\cdot\cdot} + c\,(y - x) = 0, \qquad m_2 x^{\cdot\cdot} + c\,(x - y) = 0, \quad y > 0 \qquad (3.8)$$
$$y^{\cdot +} = -R y^{\cdot -}, \qquad y = 0$$

while in the case of a kinematic constraint between mass m_1 and the fixed barrier, the motions take place in accord with the law

$$y = 0, \qquad m_2 x^{\cdot\cdot} + cx = \qquad x < 0 \qquad (3.9)$$

From the point mappings, generated by Eqs. (3.8), of surface Π into itself, we have

$$\varphi\,(\tau_0, x_0, x_0^{\cdot}, y_0^{\cdot}) \equiv (x_0^{\cdot} + \mu^{-1} R y_0^{\cdot})\,\tau_0 - (\cos\lambda\tau_0 - 1)\,x_0 - \lambda^{-1}\,(x_0^{\cdot} + R y_0^{\cdot})\sin\lambda\tau_0 = 0$$
$$\mu = \frac{m_2}{m_1}, \quad \lambda = \frac{\omega}{1+\mu}, \quad \omega^2 = \frac{c\,(m_1 + m_2)}{m_1 m_2} \qquad (3.10)$$

Using the idealization of the slipping state as a motion according to law (3.9), we obtain the following equation for seeking h:

$$\Phi\,(h, x_0, x_0^{\cdot}) \equiv \omega x_0 + x_0^{\cdot}\,\mathrm{tg}\,\omega h = 0 \qquad (3.11)$$

The system of relations (3.10), (3.11), (2.12) determine in the first approximation the boundary Γ_s. For the particular case $x_0 = 0$, considered in [5], from (3.11) we have $h = \pi/\omega$ and the equations for curve Γ_s in the form

$$x_0^{\cdot}\,(\nu - \sin\nu) - y_0^{\cdot} R\,(\nu\mu^{-1} + \sin\nu) = 0 \qquad (3.12)$$
$$\nu - \pi\,(1 - R)\,(1 - 0.4R)\,(1 + \mu)^{0.5} = 0, \qquad \nu = \lambda\tau_0$$

In the space of parameters R, μ the location of boundary C_s depends only on the velocity ratio $k = x_0^{\cdot}/y_0^{\cdot}$. In Fig. 4 the solid lines depict the boundaries of the existence regions of the slipping state, constructed by means of (3.12). Under specific initial conditions $x_0 = 0$ the quantity h is essentially large, therefore, the results obtained are subject to refinement.

We consider the second approximation. In this case, as the origin of the slipping state we take the point $M_1\,(x_1, y = 0, x_1^{\cdot}, y^{\cdot})$ corresponding to the second collision of the system with the barrier. We obtain the coordinates of point M_1 by integrating (3.8) with the initial conditions $x\,(t_0) = x_0$, $y\,(t_0) = 0$, $x^{\cdot}\,(t_0) = x_0^{\cdot}$, $y^{\cdot}\,(t_0) = -R y_0^{\cdot}$,

$$x_1 = (x_0^{\cdot} - \mu R y_0^{\cdot})\,\tau_0 + x_0, \quad x_1^{\cdot} = (1 + \mu)^{-1}\,(x_0^{\cdot} - \mu R y_0^{\cdot} + \mu\sigma) \qquad (3.13)$$
$$y_1^{\cdot} = (1 + \mu)^{-1}\,(x_0^{\cdot} - \mu R y_0^{\cdot} - \sigma), \quad \sigma = (x_0^{\cdot} + R y_0^{\cdot})\cos\lambda\tau_0 - \lambda x_0 \sin\lambda\tau_0$$

The time interval $\tau_1 = t_2 - t_1$ and the duration h are determined from the equations

$$(x_1^{\cdot} - \mu R y_1^{\cdot})\,\tau_1 - x_1\,(\cos\lambda\tau_1 - 1) - \lambda^{-1}\,(x_1^{\cdot} + R y_1^{\cdot})\sin\lambda\tau_1 = 0$$
$$\omega x_1 + x_1^{\cdot}\,\mathrm{tg}\,\omega h = 0 \qquad (3.14)$$

analogous, respectively, to conditions (3.10) and (3.11). As a result of manipulations, from relations (3.10), (3.11), (3.14) we obtain three transcendental equations

$$[\mu\,(k + R)\,(1 + R)\cos\lambda\tau_0 + (k - \mu R)\,(1 - \mu R)]\,\tau_1 - (1 + \mu)\,(k - \mu R)\,(\cos\lambda\tau_1 - 1)\,\tau_0 - \lambda^{-1}\,[(k + R)\,(\mu - R)\cos\lambda\tau_0 + (k - \mu R)\,(1 + R)]\sin\lambda\tau_1 = 0$$
$$\lambda\,(k - \mu R)\,\tau_0 - (k + R)\sin\lambda\,\tau_0 = 0 \qquad (3.15)$$
$$\mathrm{tg}\,\omega h = \frac{\mu\,(k + R)\cos\lambda\tau_0 + k - \mu R}{\omega\tau_0\,(1 + \mu)\,(\mu R - k)}$$

which, together with (2.12), determine the refined boundaries $C_s(k)$. The boundary curves of the slipping state regions, computed by formulas (3.15), (2.12), are shown in Fig. 4 by the dotted lines. Here, in addition, we have reproduced the boundary C_s for $k = 1$, obtained in [5], by a dash-dotted line. A certain difference in the locations of the boundary lines is explained by the fact that the analysis has been restricted to the second approximation, and also by the singular way that the method of [4] was applied in [5].

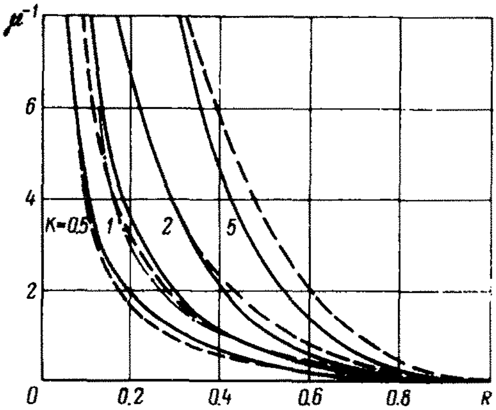

Fig. 4

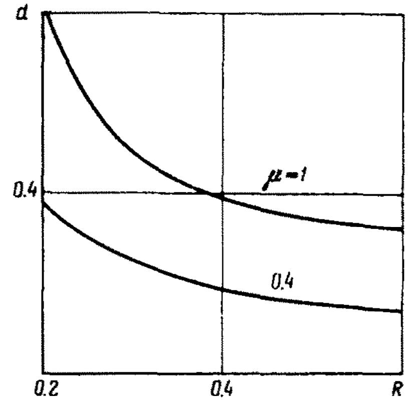

Fig. 5

4. In conclusion we consider a slipping state in a nonautonomous system with two degrees of freedom, being, in particular, a model of a shock absorber. In the resonant case the dimensionless equations of motion have the form [7]

$$x^{\cdot\cdot} + x = \sin t, \quad y^{\cdot\cdot} = -x^{\cdot\cdot}, \quad |y| < d$$
$$y^{\cdot +} = -Ry^{\cdot -}, \quad x^{\cdot +} = x^{\cdot -} + \mu(1+\mu)^{-1}(1+R)y^{\cdot -}, \quad |y| = d \tag{3.16}$$

The equations of the point mappings of the surfaces $y = \pm d$ into themselves, generated by relations (3.16), allow us to obtain the function $\varphi(\tau_0, x_0, x_0^{\cdot}, y_0^{\cdot}, t_0)$ in the form

$$x_0(1-\cos\tau_0) - \tau_0[x_0^{\cdot} + \nu^2(\mu - R)y_0^{\cdot} + {}^1/_2\cos(t_0+\tau_0)] - [x_0^{\cdot} + \mu\nu^2(1+R)y_0^{\cdot} +$$
$$+ {}^1/_2\cos t_0]\sin\tau_0 = 0, \quad \nu^{-2} = 1+\mu \tag{3.17}$$

Integrating, with the initial conditions $x(t_0) = x_0$, $x^{\cdot}(t_0) = x_0^{\cdot} + \mu\nu^2 y_0^{\cdot}$ the equations of motion of the system under a kinematic constraint between the colliding elements

$$(1+\mu)x^{\cdot\cdot} = \sin t - x; \; y = +d, \quad y^{\cdot\cdot} > 0 \quad (y = -d, \; y^{\cdot\cdot} < 0)$$

we obtain the following condition for determining the duration h:

$$\nu(\sin t_0 + \mu x_0^{\cdot})\cos\nu h + (\cos t_0 + \mu x_0^{\cdot} + \mu^2\nu^2 y_0^{\cdot})\sin\nu h + \nu^{-1}\sin(t_0+h) = 0 \tag{3.18}$$

Relations (3.17), (3.18), (2.12), obviously determine in closed form on the surfaces $y = +d$, $y = -d$ the boundaries Γ_s^+ and Γ_s^-, respectively, of the slipping motion regions Π_s^+ and Π_s^-.

To delineate, in the parameter space, the existence regions of periodic motions with a slipping state segment, it is necessary to examine the point transformation of surface Π_s^+ into surface Π_s^-. The coordinates of the fixed points corresponding to a periodic motion with an absolutely inelastic collision of the system's elements are found in the usual manner [7]. Here one of the fixed points always lies on the exit manifold M_s^+ (M_s^-) of the region Π_s^+ (Π_s^-). The coordinates of the other fixed point, in the case of simplest periodic motions with one slipping state segment as a half-period, are determined by the system of equations

$$x_0 + \sin(t_0 + h) + (\pi - h)(x_0^{\cdot} + y_0^{\cdot}) + 2d = 0$$

$$x_0 \cos h + \frac{1}{\mu}\sin(t_0 + h) + \left(x_0^{\cdot} + \frac{1}{2}\cos t_0\right)\sin h + \frac{1}{2}(\pi - t_0)\cos(t_0 + h) - \left(x_0 + \frac{1}{\mu}\sin t_0\right)\cos\nu h - \left[x_0^{\cdot} + \mu\nu(x_0^{\cdot} + y_0^{\cdot}) + \frac{1}{\mu\nu}\cos t_0\right]\sin\nu h = 0 \tag{3.19}$$

$$\left(\tfrac{1}{2}\sin t_0 - x_0\right)\sin h + x_0^{\cdot}\left[\cos h + \nu^2(\cos\nu h - \sin\nu h) + 1\right] - \tfrac{1}{2}(\pi - t_0)\sin(t_0 + h) = 0$$

$$\nu(\mu x_0 + \sin t_0)\operatorname{tg}\nu h + \left[\cos(t_0 + h) - \mu x_0^{\cdot}\right]\sec\nu h - \mu x_0^{\cdot} - \mu^2\nu^2 y_0^{\cdot} - \cos t_0 = 0$$

Relations (3.17) – (3.19), together with (2.12), impose a constraint on the parameters μ, d, R of the system being considered, defining the existence boundary of the periodic motion with a slipping state segment. Since the coordinates x_0, $x_0^{\cdot}$, $y_0^{\cdot}$ enter linearly in (3.19), the boundary surface C_s can be expressed in parametric form by three transcendental equations. Figure 5 shows the results of computation of sections of the surface C_s by planes $\mu = \text{const}$ on the interval $0.2 \leqslant R \leqslant 0.6$.

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